

# INTERSECTIONS OF AMOEBAS

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**ABSTRACT.** Amoebas are projections of complex algebraic varieties in the algebraic torus under a Log-absolute value map, which have connections to various mathematical subjects. While amoebas of hypersurfaces have been intensively studied during the last years, the non-hypersurface case is barely understood so far.

We investigate intersections of amoebas of  $n$  hypersurfaces in  $(\mathbb{C}^*)^n$ , which are genuine supersets of amoebas given by non-hypersurface varieties. Our main results are amoeba analogs of *Bernstein's Theorem* and *Bézout's Theorem* providing an upper bound for the number of connected components of such intersections. Moreover, we show that the *order map* for hypersurface amoebas can be generalized in a natural way to intersections of amoebas. We show that, analogous to the case of amoebas of hypersurfaces, the restriction of this generalized order map to a single connected component is still 1-to-1.

## 1. INTRODUCTION

Let  $I$  be an ideal generated by finitely many *Laurent polynomials*  $f_1, \dots, f_k \in \mathbb{C}[\mathbf{z}^{\pm 1}] := \mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$  and let  $\mathcal{V}(I) \subseteq (\mathbb{C}^*)^n := (\mathbb{C} \setminus \{0\})^n$  be the corresponding variety. The *amoeba*  $\mathcal{A}(I)$  of  $I$ , as originally defined by Gelfand, Kapranov, and Zelevinsky [7], is the image of  $\mathcal{V}(I)$  under the Log-absolute map given by

$$(1.1) \quad \text{Log} |\cdot| : (\mathbb{C}^*)^n \rightarrow \mathbb{R}^n, \quad (z_1, \dots, z_n) \mapsto (\log |z_1|, \dots, \log |z_n|).$$

In the special case that  $I = \langle f \rangle$ , we write  $\mathcal{A}(f)$  for simplicity.

Amoebas became prominent during the last twenty years since they provide a natural connection between algebraic geometry and tropical geometry; see [5, 12, 14] for an overview. Furthermore, amoebas are objects with a rich structure themselves and there exist connections to numerous other mathematical subjects like complex analysis [6], nonnegativity of real polynomials [9], crystal shapes [10], the topology of real curves [13], and statistical thermodynamics [17]. For an overview about amoeba theory see [4, 5, 14, 19, 22].

Though amoebas of hypersurfaces, i.e., amoebas associated to a single Laurent polynomial, have been intensively studied during the last years, the non-hypersurface case is still barely understood. For almost all properties that are true in the hypersurface case it is not known if they still hold for arbitrary varieties. One exception are statements regarding convexity, which were recently shown by Nisse and Sottile; see [16]. Another one of the few results concerning amoebas of ideals that are not principal was shown by Purbhoo. It states that the amoeba of an arbitrary ideal  $I \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$  can be written as

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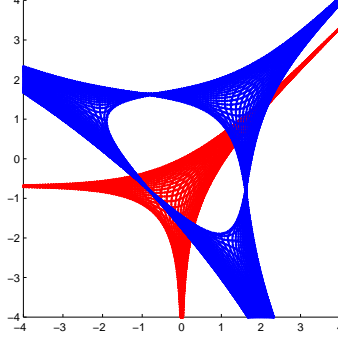


FIGURE 1. An approximation of the amoebas  $\mathcal{A}(f_1)$  and  $\mathcal{A}(f_2)$  for  $f_1 := 2z_1 + z_2 + 1$  and  $f_2 := z_1^2 z_2 + z_1 z_2^2 + 5z_1 z_2 + 1$ . The intersection  $\mathcal{A}(f_1) \cap \mathcal{A}(f_2)$  consists of two connected components.

the intersection of the amoebas of all the elements of  $I$ ; see [20, Corollary 5.6.]:

$$(1.2) \quad \mathcal{A}(I) = \bigcap_{f \in I} \mathcal{A}(f).$$

Since this result is not useful from a computational point of view, the question about the existence of an *amoeba basis* arose in the article [23] by Schroeter and the second author. Here, an amoeba basis refers to a finite set of Laurent polynomials  $f_1, \dots, f_k$  such that  $\langle f_1, \dots, f_k \rangle = I$  and  $\mathcal{A}(I) = \bigcap_{j=1}^k \mathcal{A}(f_j)$ . So far, amoeba bases are known to exist only in very special cases [23] while a recent result by Nisse claims non-existence in general [15]. By (1.2), however, the inclusion  $\mathcal{A}(I) \subseteq \bigcap_{j=1}^k \mathcal{A}(f_j)$  holds for *every* collection  $f_1, \dots, f_k$  with  $\langle f_1, \dots, f_k \rangle \subseteq I$  and it is reasonable to expect that information about  $\bigcap_{j=1}^k \mathcal{A}(f_j)$  also provides information about  $\mathcal{A}(I)$ . Phrased differently, understanding finite intersections of hypersurface amoebas is an essential interim stage for understanding amoebas of arbitrary ideals. Another key advantage of this approach is that finite intersections of hypersurface amoebas turn out to be much more accessible than amoebas of arbitrary ideals. This serves as the key motivation for this article.

Let  $\mathcal{F} := \{f_1, \dots, f_n\} \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$  be a family of  $n$ -variate complex Laurent polynomials. In this article, we study the intersection of their corresponding amoebas  $\mathcal{A}(f_1), \dots, \mathcal{A}(f_n) \subseteq \mathbb{R}^n$ .

We show that intersections  $\mathcal{I}(\mathcal{F}) := \bigcap_{j=1}^n \mathcal{A}(f_j)$  preserve a significant amount of the amoeba structure from the hypersurface case. Moreover, these intersections carry an additional interesting and rich combinatorial structure on their own.

Since, in general,  $\mathcal{I}(\mathcal{F})$  consists of several connected components, we focus on the study of the combinatorics of these components; see Figure 1 and Section 3. Particularly, the convex hull of each of these components is proven to be a simple polytope, which, in the sequel, will be referred to as *intersection polytope*. A natural question that arises is, how many connected components such an intersection  $\mathcal{I}(\mathcal{F})$  might possibly have. One of our

main results provides an upper bound for this quantity. More precisely, as an analog to the classical Bernstein's Theorem, we show the following *Amoeba Bernstein Theorem*; see Theorem 3.12 for the detailed version.

**Theorem 1.1.** *Let  $\mathcal{F} = \{f_1, \dots, f_n\} \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$  be a generic collection of Laurent polynomials. The number of connected components of  $\mathcal{I}(\mathcal{F})$  is bounded from above by the mixed volume  $\text{MV}(\text{New}(f_1), \dots, \text{New}(f_n))$ .*

We specify in Section 3, what we understand by a “generic collection of Laurent polynomials”. We want to remark that this theorem is highly non-obvious since it is unclear whether every connected component of  $\mathcal{I}(\mathcal{F})$  contains a point, which is the projection of a point in  $\mathcal{V}(\langle f_1, \dots, f_n \rangle)$  with respect to the  $\text{Log}|\cdot|$ -map. As an immediate consequence we obtain the following *Amoeba Bézout Theorem*; see Theorem 3.13 for the detailed version.

**Theorem 1.2.** *Let  $\mathcal{F} = \{f_1, \dots, f_n\} \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$  be a generic family of Laurent polynomials. The number of connected components of  $\mathcal{I}(\mathcal{F})$  is smaller than or equal to the product of the total degrees of the  $f_j$ .*

In Section 4, we construct a generalization of the *order map* of hypersurface amoebas to our setting. The usual order map, introduced by Forsberg, Passare, and Tsikh [6], relates the components of the complement of a hypersurface amoeba  $\mathcal{A}(f)$  to the lattice points in the corresponding Newton polytope  $\text{New}(f)$ ; see Section 2.2 for details. Given a collection  $\mathcal{F}$  of  $n$  Laurent polynomials, we define a natural generalization of the order map to the vertex sets of the intersection polytopes of  $\mathcal{F}$  and thereby also to the polytope, which is the convex hull of the entire intersection  $\mathcal{I}(\mathcal{F})$ . We show that several properties of the order map for the hypersurface case are preserved in this more general setting; see Theorems 4.1 and 4.4, as well as Corollary 4.5. The next theorem summarizes those results. The notion of a *mixed normal cone*, which is used in this statement, will be explained in Section 4.

**Theorem 1.3.** *Let  $\mathcal{F} := \{f_1, \dots, f_n\} \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$  be a generic family of Laurent polynomials. Let  $K$  be a connected component of  $\mathcal{I}(\mathcal{F})$ , let  $P_K := \text{conv}(K)$  be the corresponding intersection polytope and let  $P := \text{conv}(\mathcal{I}(\mathcal{F}))$ . Then the following hold:*

- (a) *There exists a generalized order map from the vertices of  $P_K$  and  $P$ , respectively, to  $(\text{New}(f_1) \times \dots \times \text{New}(f_n)) \cap \mathbb{Z}^{n \times n}$ , that is injective on  $P_K$  and on  $P$ , respectively.*
- (b) *The vertices of  $P$  are in 1-to-1-correspondence with those vertices of the Minkowski sum  $\text{New}(f_1) + \dots + \text{New}(f_n)$ , which have a mixed normal cone.*

The article is organized as follows. In Section 2 we fix our notation and introduce background information about amoebas, particularly about the order map and the spine. Moreover, we provide a short review of mixed volumes as well as the classical and tropical versions of Bernstein's Theorem. In Section 3 we discuss combinatorial properties of intersections of hypersurface amoebas and their associated intersection polytopes. The *Amoeba Bernstein Theorem* is the main result of this section. In Section 4 we define the generalized order map and discuss its properties, including the injectivity statements mentioned above.

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## 2. PRELIMINARIES

**2.1. Polytopes and their Normal Fans.** Throughout this article, we assume that the reader is familiar with basic objects from discrete geometry. For background information we recommend [25] as a reference.

In the following, we introduce notions of some subjects that are less well-known and of particular importance for this article. Given a Laurent polynomial  $f := \sum_{a \in \mathbb{Z}^n} \lambda_a \mathbf{z}^a \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ , where  $\mathbf{z}^a := z_1^{a_1} \cdots z_n^{a_n}$  for  $a := (a_1, \dots, a_n) \in \mathbb{Z}^n$ , its *Newton polytope*  $\text{New}(f)$  is the lattice polytope defined by

$$\text{New}(f) := \text{conv}\{a \in \mathbb{Z}^n : \lambda_a \neq 0\} \subseteq \mathbb{R}^n.$$

In the following, the set  $\{a : \lambda_a \neq 0\}$  will also be referred to as the *support* of  $f$ .

Given a polytope  $P \subseteq \mathbb{R}^n$ , we are frequently interested in its normal fan, whose definition we recall now. To every non-empty face  $G \in P$  one associates the following *normal cone*

$$\text{NF}_G(P) := \{\mathbf{c} \in \mathbb{R}^n : G \subseteq \{\mathbf{y} \in P : \langle \mathbf{c}, \mathbf{y} \rangle = \max_{\mathbf{w} \in P} \langle \mathbf{c}, \mathbf{w} \rangle\}\},$$

where  $\langle \cdot, \cdot \rangle$  denotes the standard scalar product in  $\mathbb{R}^n$ . The collection of these cones is called the *normal fan* of  $P$ , denoted by  $\text{NF}(P)$ :

$$\text{NF}(P) := \{\text{NF}_G(P) : \emptyset \neq G \subseteq P \text{ face of } P\}.$$

If  $P := \text{New}(f)$  is the Newton polytope of a Laurent polynomial  $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ , then we write  $\text{NF}_G(f)$  for  $\text{NF}_G(\text{New}(f))$ , where  $G$  is a face of  $\text{New}(f)$ . Similarly, we write  $\text{NF}(f)$  for the normal fan of  $\text{New}(f)$ .

**2.2. The Order Map.** Given a Laurent polynomial  $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ , the complement of its amoeba  $\mathcal{A}(f)$  consists of several bounded or unbounded connected components. As shown in [6], there exists a close connection between those components and lattice points in the corresponding Newton polytope  $\text{New}(f)$ . The precise relation is given via the *order map* introduced by Forsberg, Passare, and Tsikh [6]:

$$(2.1) \quad \text{ord} : \mathbb{R}^n \setminus \mathcal{A}(f) \rightarrow \mathbb{R}^n, \quad \mathbf{w} \mapsto (u_1, \dots, u_n) \text{ with} \\ u_j := \frac{1}{(2\pi i)^n} \int_{\text{Log } |\mathbf{z}|=\mathbf{w}} \frac{z_j \partial_j f(\mathbf{z})}{f(\mathbf{z})} \frac{dz_1 \cdots dz_n}{z_1 \cdots z_n} \text{ for all } 1 \leq j \leq n.$$

In the following, slightly abusing notation, we write  $\text{im}(\text{ord}(f))$  for the image of  $\mathbb{R}^n \setminus \mathcal{A}(f)$  under the order map.

The order map can be understood as a multivariate analogue of the classical *argument principle* from complex analysis. The following theorem by Forsberg, Passare and Tsikh [6] provides the indicated connection between components of the complement of  $\mathcal{A}(f)$  and lattice points in  $\text{New}(f)$ .

**Theorem 2.1.** *Given a Laurent polynomial  $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$  the image of the order map  $\text{im}(\text{ord}(f))$  is contained in  $\text{New}(f) \cap \mathbb{Z}^n$ . If  $\mathbf{w}, \mathbf{w}' \in \mathbb{R}^n \setminus \mathcal{A}(f)$ , then  $\mathbf{w}$  and  $\mathbf{w}'$  belong to the same component of the complement of  $\mathcal{A}(f)$  if and only if  $\text{ord}(\mathbf{w}) = \text{ord}(\mathbf{w}')$ .*

As a consequence of the previous theorem, every component of the complement of a given amoeba  $\mathcal{A}(f)$  corresponds to a unique lattice point in the Newton polytope  $\text{New}(f)$  of  $f$ . In the following, we denote for each  $\alpha \in \text{New}(f) \cap \mathbb{Z}^n$  its corresponding (possibly empty) component of the complement of  $\mathcal{A}(f)$  by  $E_\alpha(f)$ , i.e.,

$$E_\alpha(f) := \{\mathbf{w} \in \mathbb{R}^n \setminus \mathcal{A}(f) : \text{ord}(\mathbf{w}) = \alpha\}.$$

Points in  $E_\alpha(f)$  are said to be of *order  $\alpha$*  and  $E_\alpha(f)$  is called *the component of order  $\alpha$*  of the complement.

Besides the just described connection between components of the complement of  $\mathcal{A}(f)$  and lattice points in  $\text{New}(f)$ , Gelfand, Kapranov and Zelevinsky [7] showed that the former ones are also strongly linked to the normal fan of  $\text{New}(f)$ . Indeed, the precise relation is the following:

**Theorem 2.2.** *Let  $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$  with support set  $A \subseteq \mathbb{Z}^n$ . The set of vertices of  $\text{New}(f)$  is in bijective correspondence with a subset of the components of the complement of  $\mathcal{A}(f)$ . Namely, let  $\alpha \in A$  be a vertex in  $\text{New}(f)$  and  $\text{NF}_\alpha(f)$  be the corresponding cone in  $\text{NF}(f)$ . Then there exists a unique non-empty, unbounded component  $E_\alpha(f)$  in the complement of  $\mathcal{A}(f)$  that contains an affine translation of  $\text{NF}_\alpha(f)$ .*

**2.3. Tropical Geometry and the Spine.** Looking at the image of an amoeba, one immediately observes that it has finitely many “tentacles”, which point in different directions. These tentacles direct to a set of points at infinity, which is called the *logarithmic limit set* and was introduced by Bergman [1]. For a Laurent polynomial  $f$  with amoeba  $\mathcal{A}(f)$  and any positive real number  $r \in \mathbb{R}$ , one defines

$$(2.2) \quad \mathcal{A}_r(f) := (1/r \cdot \mathcal{A}(f)) \cap \mathbb{S}^n.$$

Here,  $1/r \cdot \mathcal{A}(f) := \{1/r \cdot \mathbf{w} : \mathbf{w} \in \mathcal{A}(f)\}$  and  $\mathbb{S}^n$  denotes the  $n$ -dimensional unit sphere  $\mathbb{S}^n := \{\mathbf{w} \in \mathbb{R}^n : \|\mathbf{w}\|_2 = 1\}$ . The *logarithmic limit set*  $\mathcal{A}_\infty(f)$  is defined as

$$\mathcal{A}_\infty(f) := \lim_{r \rightarrow \infty} \mathcal{A}_r(f).$$

It was shown by Bieri and Groves [3] that  $\mathcal{A}_\infty(f)$  is a rational, polyhedral fan on the unit sphere; see also [12]. Note also that amoebas  $\mathcal{A}(f)$  are unbounded, and their complements are open. This fact will be used in Section 2.3.

In the following, we introduce the *spine* of an amoeba  $\mathcal{A}(f)$ , which is a more sophisticated way to describe the latter one. In Section 2.3, we will repeatedly use that the spine is both, a piecewise linear deformation retract of  $\mathcal{A}(f)$  and a tropical hypersurface.

First, we need to recall some notion from tropical geometry. For additional background on tropical geometry we refer to [12]. The *tropical semiring*  $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$  is defined by the operations

$$a \oplus b := \max\{a, b\}, \quad \text{and} \quad a \odot b := a + b.$$

Note that  $-\infty$  is the neutral element for the tropical addition  $\oplus$ . Note that some authors prefer the minimum together with  $+\infty$  instead of the maximum as tropical addition. A *tropical monomial* is a function

$$(\mathbb{R} \cup \{-\infty\})^n \rightarrow \mathbb{R} \cup \{-\infty\}, \quad (z_1, \dots, z_n) \mapsto b_a \odot \mathbf{z}^a := b_a \odot z_1^{a_1} \odot \dots \odot z_n^{a_n}$$

with  $b_a \in \mathbb{R}$  and  $a := (a_1, \dots, a_n) \in \mathbb{N}^n$ . In terms of classical operations a tropical monomial is the affine linear form  $b_a + \langle \mathbf{z}, a \rangle$ . A *tropical polynomial* with *support set*  $A \subseteq \mathbb{N}^n$  is a finite tropical sum of tropical monomials, i.e., it is a function

$$(\mathbb{R} \cup \{-\infty\})^n \rightarrow \mathbb{R} \cup \{-\infty\}, \quad (z_1, \dots, z_n) \mapsto \bigoplus_{a \in A} b_a \odot \mathbf{z}^a := \max_{a \in A} \{b_a + \langle \mathbf{z}, a \rangle\},$$

where  $b_a \in \mathbb{R} \setminus \{0\}$ . For a tropical polynomial  $h$  as above, its *tropical hypersurface* or *tropical variety*  $\mathcal{T}(h)$  is defined as the set of points  $\mathbf{x}$  in  $(\mathbb{R} \cup \{-\infty\})^n$  such that the maximum of  $\{b_a + \langle \mathbf{x}, a \rangle : a \in A\}$  is attained at least twice. The tropical hypersurface  $\mathcal{T}(h)$  is a polyhedral complex, which is dual to a (regular) subdivision of the Newton polytope of  $h$ .

The definition of the spine of an amoeba requires the definition of the *Ronkin function* [21]; see also [18]. Let  $\Omega$  be a convex open set in  $\mathbb{R}^n$  and let  $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ , which is defined on  $\text{Log}^{-1}|\Omega|$ . The *Ronkin function*  $R_f$  is defined by the integral

$$R_f : \Omega \rightarrow \mathbb{R}, \quad \mathbf{x} \mapsto \frac{1}{(2\pi i)^n} \int_{\text{Log}^{-1}|\mathbf{x}|} \frac{\log |f(\mathbf{z})| dz_1 \dots dz_n}{z_1 \dots z_n}.$$

The next theorem collects some important properties of  $R_f$ .

**Theorem 2.3** (Ronkin [21] / Passare, Rullgård [18]). *Let  $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$  be a holomorphic function. Then  $R_f$  is a convex function. If  $U \subseteq \Omega$  is a connected open set, then the restriction of  $R_f$  to  $U$  is affine linear if and only if  $U \cap \mathcal{A}(f) = \emptyset$ . If  $\mathbf{x}$  is in the complement of  $\mathcal{A}(f)$ , then the gradient of  $R_f(\mathbf{x})$  equals the order of  $\mathbf{x}$ .*

Given a Laurent polynomial  $f$  and a point  $\alpha$  in its support set  $A$ , we have seen in Section 2.2 that the corresponding component  $E_\alpha(f)$  of the complement of  $\mathcal{A}(f)$  is non-empty if and only if  $\alpha \in \text{im}(\text{ord}(f))$ . For every  $\alpha \in \text{im}(\text{ord}(f))$  one defines the *Ronkin coefficient* of  $\alpha$  by

$$(2.3) \quad r_\alpha := R_f(\mathbf{x}) - \langle \alpha, \mathbf{x} \rangle \quad \text{for every } \mathbf{x} \in E_\alpha(f).$$

Note that, due to Theorem 2.3,  $r_\alpha$  is well-defined. The Ronkin coefficients give rise to the following tropical polynomial

$$(2.4) \quad \text{SpineT}(f) := \bigoplus_{\alpha \in \text{im}(\text{ord}(f))} r_\alpha \oplus \mathbf{x}^\alpha.$$

Finally, the tropical hypersurface given by  $\text{SpineT}(f)$  is called the *spine* of  $f$ , which is denoted by  $\mathcal{S}(f)$ :

$$\mathcal{S}(f) := \mathcal{T}(\text{SpineT}(f)).$$

It was shown by Passare and Rullgård [18] that the spine of a Laurent polynomial  $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$  is a deformation retract of  $\mathcal{A}(f)$ .

For additional background about the spine and the relation between amoebas and tropicalizations we recommend [5, 19]

**2.4. Mixed Volumes and the Bernstein Theorem.** Recall that the *Minkowski sum* of  $n$ -polytopes  $P$  and  $Q$  is defined as (see [25, p. 28]):

$$P + Q := \{\mathbf{p} + \mathbf{q} : \mathbf{p} \in P, \mathbf{q} \in Q\}.$$

Following [7, p. 205] we provide the following definition of *mixed volumes*. Let  $P_1, \dots, P_n$  be arbitrary (convex) polytopes in  $\mathbb{R}^n$  and let  $\lambda P_i$  for  $\lambda \in \mathbb{R}$  be the scaled version of  $P_i$ , that is  $\lambda P_i := \{\lambda \mathbf{p} : \mathbf{p} \in P_i\}$ . Given a translation-invariant volume form  $\text{Vol}$  on  $\mathbb{R}^n$ , the expression

$$(2.5) \quad \text{Vol}(\lambda_1 P_1 + \dots + \lambda_n P_n)$$

is a homogeneous polynomial in  $\lambda_1, \dots, \lambda_n$  of degree  $n$ .

**Definition 2.4.** The *mixed volume*  $\text{Vol}(P_1, \dots, P_n)$  is the coefficient of the monomial  $\lambda_1 \dots \lambda_n$  in the polynomial (2.5). More explicitly, we have

$$\text{Vol}(P_1, \dots, P_n) := \frac{1}{n!} \sum_{j=1}^n (-1)^{n-j} \sum_{1 \leq i_1 < \dots < i_j \leq n} \text{Vol}(P_{i_1} + \dots + P_{i_j}).$$

Since in the sequel, we will always consider lattice polytopes, we choose **MV** as the volume form that is induced by the lattice  $\mathbb{Z}^n$  and that satisfies that every standard simplex has volume 1; see [7, Chapter 5, Section 3D]. The classical Bernstein Theorem states the following; see [2] and [7, Theorem 2.8., p. 206]:

**Theorem 2.5** (Bernstein Theorem). *Let  $A_1, \dots, A_n \subset \mathbb{Z}^n$  be finite sets such that their union generates  $\mathbb{Z}^n$  as an affine lattice. Let  $P_i \subseteq \mathbb{R}^n$  be the convex hull of  $A_i$ , and let  $\mathbb{C}^{A_i}$  be the space of Laurent polynomials in  $z_1, \dots, z_n$  with support set  $A_i$ . Then there exists a dense Zariski open subset  $U \subseteq \mathbb{C}^{A_1} \times \dots \times \mathbb{C}^{A_n}$  with the following property: for any  $(f_1, \dots, f_n) \in U$ , the number of solutions of the system of equations  $f_1(\mathbf{z}) = \dots = f_n(\mathbf{z}) = 0$  in  $(\mathbb{C}^*)^n$  equals the mixed volume  $\text{MV}(P_1, \dots, P_n)$ .*

Let  $P_1, \dots, P_n$  be lattice  $n$ -polytopes as before and let  $P := P_1 + \dots + P_n$ . A sum  $C := F_1 + \dots + F_n$ , where  $F_i \subseteq P_i$  is a face ( $1 \leq i \leq n$ ), is called a *cell* of  $P$ . A *subdivision* of  $P$  is a collection  $\Gamma := \{C_1, \dots, C_m\}$  of cells such that each cell is of full dimension, the intersection of two cells is a face of both and the union of all cells covers  $P$ . A subdivision  $\Gamma$  is called *mixed* if for each cell  $C = F_1 + \dots + F_n \in \Gamma$  one has  $n = \dim F_1 + \dots + \dim F_n$ . A cell  $C$  is called *mixed* if every  $P_i$  contributes with a face of dimension at least 1 to  $C$ . For further information and the following statement see [24]; see also [8] and [12, Section 4.6.]

**Lemma 2.6.** *Let  $P_1, \dots, P_n$  be lattice  $n$ -polytopes and let  $P = P_1 + \dots + P_n$  equipped with an arbitrary mixed subdivision  $\Gamma$ . Then we have*

$$\text{MV}(P_1, \dots, P_n) = \sum_{C \text{ mixed cell in } \Gamma} \text{MV}(C).$$



An intersection of  $k$  tropical hypersurfaces in  $\mathbb{R}^n$  is called *proper* if it has codimension  $n - k$ . Any  $\varepsilon$ -perturbation of an arbitrary intersection of  $k$  tropical hypersurfaces becomes transversal. Give  $k$  tropical hypersurfaces their *stable* intersection is defined as the limit, for  $\varepsilon \rightarrow 0$ , of an  $\varepsilon$ -perturbation of the original (possibly non-transversal) intersection. Every stable intersection is proper; see [24, Section 3.1.] and [12, Section 3.6.] for further details.

There exists a tropical analog of the Bernstein Theorem. We provide a short version here, which is sufficient for our needs. For more information about the tropical Bernstein theorem including the detailed version see [12, Theorem 4.6.9., p. 196].

**Theorem 2.7** (Tropical Bernstein Theorem). *Let  $\mathcal{T}(h_1), \dots, \mathcal{T}(h_n) \in \mathbb{R}^n$  be generic tropical hypersurfaces which are dual to regular subdivisions of  $\text{New}(h_1), \dots, \text{New}(h_n) \subseteq \mathbb{R}^n$ . The multiplicity of each point  $\mathbf{w}$  in the stable intersection  $\mathcal{T}(h_1) \cap \dots \cap \mathcal{T}(h_n)$  equals the mixed volume  $\text{MV}(C)$  where  $C$  is the mixed cell in the subdivision of  $\text{New}(h_1) + \dots + \text{New}(h_n)$  induced by  $\mathcal{T}(h_1) \cap \dots \cap \mathcal{T}(h_n)$  corresponding to  $\mathbf{w}$ .*

### 3. COMBINATORICS OF INTERSECTIONS OF AMOEBAS

The aim of this section is to study basic combinatorial properties of intersections of amoebas of hypersurfaces.

We start by fixing some notation that will be used during this and the next section. In the following, we will always assume that  $n \geq 2$ . Throughout this article, we will call a family  $\mathcal{F} := \{f_1, \dots, f_n\} \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$  of Laurent polynomials *generic* if the following conditions hold:

- (1) every  $f_j \in \mathcal{F}$  is irreducible,
- (2) for every  $1 \leq k \leq n$  and every  $k$ -element subset  $\{f_{i_1}, \dots, f_{i_k}\} \subseteq \mathcal{F}$  the intersection  $\bigcap_{\ell=1}^k \partial(\mathcal{A}(f_{i_\ell}))$  has codimension  $k$ ,
- (3) for every  $1 \leq k \leq n$  and every  $k$ -element subset  $\{f_{i_1}, \dots, f_{i_k}\} \subseteq \mathcal{F}$  the intersection of the corresponding spines  $\mathcal{S}(f_{i_1}), \dots, \mathcal{S}(f_{i_k})$  is stable and hence of codimension  $k$ .
- (4) for every  $1 \leq j \leq n$  and for every  $p \in \partial(\mathcal{A}(f_j))$ , there exists a unique component of the complement of  $\mathcal{A}(f_j)$  containing  $p$  in its closure.

Moreover, a single Laurent polynomial  $f$  is *generic*, if it satisfies condition (1) and (4).

Note that this kind of genericity implies several consequences, which will be crucial in some of the later proofs. In particular, the boundaries of all amoebas in  $\mathcal{F}$  have a non-trivial intersection. Thus, the boundaries of any two amoebas intersect in codimension 2. Hence, it cannot happen that  $\mathcal{A}(f_i) \subseteq \mathcal{A}(f_j)$  for some  $1 \leq i, j \leq n$ , where  $i \neq j$ .

We remark that some of the genericity assumptions for some of the later statements are redundant if one considers compactified amoebas instead of usual amoebas. Compactified amoebas are obtained via the toric moment map; see [7, p. 198 et seq.], see also [14]. Since the assumptions for genericity are minor, we work with usual amoebas to keep the article accessible for a broader audience.

In the following, we consider a generic family  $\mathcal{F} := \{f_1, \dots, f_n\} \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$  of Laurent polynomials, and we let  $\mathcal{A}(f_1), \dots, \mathcal{A}(f_n)$  be the amoebas corresponding to  $f_1, \dots, f_n$ . We



are interested in the intersection  $\mathcal{I}(\mathcal{F}) := \bigcap_{j=1}^n \mathcal{A}(f_j)$ . An initial example shows that, in general,  $\mathcal{I}(\mathcal{F})$  is disconnected.

**Example 3.1.** Let  $\mathcal{F} := \{f_1, f_2\} \subseteq \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}]$ , where  $f_1(z_1, z_2) := 2z_1 + z_2 + 1$  and  $f_2(z_1, z_2) := z_1^2 z_2 + z_1 z_2^2 + 5z_1 z_2 + 1$ . One can see from Figure 1 that  $\mathcal{A}(f_1) \cap \mathcal{A}(f_2)$  consists of two disjoint connected components.

We investigate basic properties of the connected components of  $\mathcal{I}(\mathcal{F})$ . In order to do so, we need the following lemma concerning strictly convex sets.

**Lemma 3.2.** Let  $n \geq 2$  and let  $B \subseteq \mathbb{R}^n$  be an  $n$ -dimensional closed ball. Let  $s \geq 2$  and let  $L_1, \dots, L_s \subseteq \mathbb{R}^n$  be  $n$ -dimensional strictly convex sets satisfying the following conditions:

- (a)  $\bigcap_{i=1}^s L_i \cap B^\circ$  is non-empty of dimension smaller than or equal to  $n - 1$ , where  $B^\circ$  denotes the interior of  $B$ .
- (b)  $B \subseteq \bigcup_{i=1}^s L_i$ .

Then  $s \geq n + 1$ .

*Proof.* We prove the statement by induction on  $n$ . For  $n = 2$ , assume, by contradiction, that there exist two strictly convex sets  $L_1, L_2$  satisfying conditions (a) and (b). Since  $L_1$  and  $L_2$  are strictly convex and since  $L_1 \cap L_2 \cap B$  is convex, (a) implies that  $L_1 \cap L_2 \cap B$  equals a single point  $v$ . Since  $L_1$  and  $L_2$  are both strictly convex, there exists a line  $L$  passing through  $v$  such that  $L_i \cap L = \{p\}$  for  $1 \leq i \leq 2$  and  $L_1$  and  $L_2$  lie on different sides of  $L$ . But then  $\emptyset \neq (L \setminus \{p\}) \cap B \not\subseteq L_1 \cup L_2$ , which contradicts (b).

Now assume,  $n \geq 3$ . Let  $s \geq 2$  and  $L_1, \dots, L_s$  strictly convex such that (a) and (b) hold. As for  $n = 2$ , strict convexity implies that  $\bigcap_{i=1}^s L_i \cap B^\circ$  consists of a single point  $v$ . Let  $H$  be a hyperplane in  $\mathbb{R}^n$  containing  $v$ . Then  $H \cap L_1, \dots, H \cap L_s$  yields a covering of the  $(n - 1)$ -dimensional ball  $B \cap H$  with strictly convex sets such that (a) and (b) hold. By induction we get  $s \geq n$ . Now consider an  $i$  such that  $H \cap L_i$  is of dimension  $n - 1$ . Such an  $L_i$  exists due to (b). Let  $G \subseteq \mathbb{R}^n$  be a hyperplane such that  $L_i$  lies on one side of  $G$ , intersecting  $G$  only in  $v$ . Such a  $G$  exists due to the strict convexity of  $L_i$ . Applying induction to the  $(n - 1)$ -dimensional ball  $B \cap G$ , we know that there are at least  $n$  strictly convex sets necessary to cover  $B \cap G$ . Since  $L_i$  does intersect  $B \cap G$  only in  $v$ , we conclude, that there exist at least  $n$  strictly convex sets in the collection  $L_1, \dots, L_s$  different from  $L_i$ . Therefore, we have  $s \geq n + 1$ .  $\square$

The next theorem states some basic properties of the connected components of  $\mathcal{I}(\mathcal{F})$ .

**Theorem 3.3.** Let  $n \geq 2$  and let  $\mathcal{F} := \{f_1, \dots, f_n\} \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$  be generic. Then:

- (a) Every connected component of  $\mathcal{I}(\mathcal{F})$  is a bounded and closed set and, as such, particularly compact.
- (b) Every connected component of  $\mathcal{I}(\mathcal{F})$  has dimension  $n$ .

*Proof.* (a) Consider a connected component  $K$  of  $\mathcal{I}(\mathcal{F})$ . Recall the definition of  $\mathcal{A}_r(f)$  from (2.2). Assume by contradiction that  $K$  is unbounded. Then we have particularly for every  $r \in \mathbb{R}_{\geq 1}$  that  $\bigcap_{j=1}^n \mathcal{A}_r(f_j) \neq \emptyset$ . Thus,  $\lim_{r \rightarrow \infty} \bigcap_{j=1}^n \mathcal{A}_r(f_j) \neq \emptyset$  and hence

$\bigcap_{j=1}^n \mathcal{A}_\infty(f_j) \neq \emptyset$ . This means that the intersection of the spines  $\mathcal{S}(f_1), \dots, \mathcal{S}(f_n)$  is of codimension smaller than  $n$ , which contradicts assumption (3) in our definition of generic.

Let  $B_r(\mathbf{0})$  denote the closed  $n$ -ball of radius  $r$  around the origin. Since  $K$  is bounded there exists an  $r$  such that  $K$  is a connected component of  $B_r(\mathbf{0}) \cap \bigcap_{j=1}^n \mathcal{A}(f_j)$ . Recall from Section 2.3 that the components of the complement of amoebas are open. Therefore,  $B_r(\mathbf{0}) \cap \bigcap_{j=1}^n \mathcal{A}(f_j)$  is an intersection of closed sets. Thus,  $K$  is closed and hence compact.

(b) Given a subset  $J \subseteq [n] := \{1, \dots, n\}$ , we set  $\mathcal{F}_J := \{f_j : j \in J\}$  and  $\mathcal{I}(\mathcal{F}_J) := \bigcap_{j \in J} \mathcal{A}(f_j)$ . We show by induction on  $\#J$  that for all  $J \subseteq [n]$  any connected component of  $\mathcal{I}(\mathcal{F}_J)$  has dimension  $n$ .

Since  $n \geq 2$ , the statement is clear for  $\#J = 1$ .

Let  $\#J = m \geq 2$ . Without loss of generality let  $J = [m]$ . By contradiction, assume that  $K$  is a connected component of  $\mathcal{I}(\mathcal{F}_{[m]})$  such that  $\dim K < n$ . First, we show that  $K$  has to be contained in the boundary of  $\mathcal{A}(f_j)$  for  $1 \leq j \leq m$ , in this case. Suppose without loss of generality that  $K \subseteq \mathcal{A}(f_m) \setminus \partial(\mathcal{A}(f_m))$ . Consider a point  $p$  in the relative interior of  $K$ . Since  $p$  lies in the interior of  $\mathcal{A}(f_m)$ , there exists a small  $n$ -dimensional neighborhood  $U$  of  $p$ , such that  $K \cap U = \mathcal{I}(\mathcal{F}_{[m-1]}) \cap U$ . By induction, it follows that  $\mathcal{I}(\mathcal{F}_{[m-1]}) \cap U$  is  $n$ -dimensional, hence a contradiction to the assumption that  $\dim(K) < n$ .

Being  $K$  contained in  $\partial(\mathcal{A}(f_j))$ ,  $1 \leq j \leq m$ , means that  $K$  lies in the intersection of the boundaries of specific components  $E_{\alpha_j}(f_j)$  of the complement of  $\mathcal{A}(f_j)$  for  $1 \leq j \leq m$ .

Hence, if  $q$  is a point in the relative interior of  $K$ , then there exists a small  $n$ -dimensional neighborhood  $V$  of  $q$  such that  $\left(\bigcup_{j=1}^m \overline{E_{\alpha_j}(f_j)}\right) \cap V = V$ . As  $2 \leq m \leq n < n+1$ , this again yields a contradiction to Lemma 3.2.  $\square$

In order to obtain information about the connected components of the intersection  $\mathcal{I}(\mathcal{F})$  of  $\mathcal{A}(f_1), \dots, \mathcal{A}(f_n)$ , it is important to understand the boundaries of those components. Clearly, they can be described by means of intersections of boundaries of specific components of the complements of subcollections of  $\mathcal{A}(f_1), \dots, \mathcal{A}(f_n)$ . The following simple lemma is a direct consequence of our notion of genericity, defined at the beginning of this section.

**Lemma 3.4.** *Let  $\mathcal{F} := \{f_1, \dots, f_k\} \subset \mathbb{C}[\mathbf{z}^{\pm 1}]$  be a generic collection of Laurent polynomials. Then  $\bigcap_{j=1}^k \partial(\mathcal{A}(f_j) \setminus \mathcal{A}_\infty(f_j))$  is of dimension  $n - k$ . Particularly, if  $k = n$ , then the intersection is zero dimensional and of finite cardinality.*

Note that it is immediate from Lemma 3.4, that in our situation the intersection of the boundaries of  $\mathcal{A}(f_1), \dots, \mathcal{A}(f_n)$  is always a finite point set. This motivates the following definition.

**Definition 3.5.** *Let  $\mathcal{F} := \{f_1, \dots, f_n\} \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$  be a generic collection of Laurent polynomials.*

- (a)  $\bigcap_{j=1}^n \partial(\mathcal{A}(f_j))$  is called the set of **vertices** of  $\mathcal{I}(\mathcal{F})$ , denoted by  $V(\mathcal{F})$ .
- (b) For a connected component  $K$  of  $\mathcal{I}(\mathcal{F})$  we call  $V(K) := K \cap V(\mathcal{F})$  the set of **vertices** of  $K$ .

By Definition 3.5 vertices of a connected component  $K$  of  $\mathcal{I}(\mathcal{F})$  lie on the boundary of  $K$ . More generally, we can decompose the boundary of  $K$  as the union of disjoint

open pieces, each of which is contained in the intersection of finitely many  $\overline{\partial E_{\alpha_j}(f_j)}$ . This motivates the following definition of  $k$ -faces of an  $K$ .

**Definition 3.6.** Let  $\mathcal{F} := \{f_1, \dots, f_n\} \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$  be a generic collection of Laurent polynomials. Let  $K$  be a connected component of  $\mathcal{I}(\mathcal{F})$ . Let  $0 \leq k \leq n - 1$ .

- (a) A non-empty and connected subset  $F \subsetneq K$  is called a  **$k$ -face** of  $K$ , if there exist unique  $E_{\alpha_1}(f_{j_1}), \dots, E_{\alpha_{n-k}}(f_{j_{n-k}})$  such that  $\mathbf{x} \in \bigcap_{s=1}^{n-k} \overline{\partial E_{\alpha_s}(f_{j_s})}$  for all  $\mathbf{x} \in F$ .
- (b) An  $(n - 1)$ -dimensional face of  $K$  is called a **facet** of  $K$ .

We remark that Definition 3.5 combined with our definition of genericity imply the existence of a *face lattice* for every connected component  $K$  of  $\mathcal{I}(\mathcal{F})$ , where, as usual, faces are ordered by inclusion.

Note that a priori the definition of a face does not exclude that a single amoeba contributes with multiple components of its complement to an intersection that describes a specific face. The following lemma, however, shows that this case can never occur.

**Lemma 3.7.** Let  $\mathcal{F} := \{f_1, \dots, f_n\} \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$  be generic. Let  $K$  be a connected component of  $\mathcal{I}(\mathcal{F})$ . Let  $F$  be a  $k$ -face of  $K$  that is given by  $E_{\alpha(l_1)}(f_{j_1}), \dots, E_{\alpha(l_{n-k})}(f_{j_{n-k}})$ . Then all the  $f_{j_s}$  are distinct.

*Proof.* It suffices to observe that condition (4) in the definition of genericity implies that for every  $f_i$  and different non-empty components of the complement  $E_{\alpha}(f_i)$  and  $E_{\beta}(f_i)$  of  $\mathcal{A}(f_i)$ , one has  $\overline{E_{\alpha}(f_i)} \cap \overline{E_{\beta}(f_i)} = \emptyset$ .  $\square$

Given a connected component  $K$  of  $\mathcal{I}(\mathcal{F})$  with set of vertices  $V(K)$ , we define the polytope  $P_K$  as the convex hull of  $V(K)$ . All polytopes arising in this way will be called **intersection polytopes** of  $\mathcal{F}$ . We remark that though the vertex set of an intersection polytope  $P_K$  is clearly contained in  $V(K)$ , it does not have to coincide with  $V(K)$ ; see Figure 2. In the following, we use  $V(P_K)$  to denote the vertex set of  $P_K$ .

The following theorem shows that the vertex set  $V(P_K)$  of  $P_K$  coincides with the set of extreme points of  $\text{conv}(K)$ .

**Theorem 3.8.** Let  $\mathcal{F} := \{f_1, \dots, f_n\} \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$  be generic and let  $K$  be a connected component of  $\mathcal{I}(\mathcal{F})$ . Then the vertex set  $V(P_K)$  of  $P_K$  is given by the set of extreme points of  $\text{conv}(K)$ . Particularly, we have  $\text{conv}(K) = P_K$ .

*Proof.* Let  $E(K)$  be the set of extreme points of  $\text{conv}(K)$ . First, we show that  $E(K) \subseteq V(K)$ . Let  $p \in \text{conv}(K)$  be an extreme point.  $p$  has to lie on the boundary of  $K$  and hence there exists a face  $F$  of  $K$  such that  $p \in F$ . Let  $F$  be the smallest such face of  $K$  with respect to inclusion and dimension and let  $k = \dim F$ . Without loss of generality (after renumbering), suppose that  $F$  can be described by  $\bigcap_{j=1}^k \overline{\partial E_{\alpha_j}(f_j)}$ . If  $k = n$ , then by definition of vertices of  $K$ , it already follows that  $p$  itself is a vertex of  $K$ . So, assume  $k < n$ . For  $\varepsilon > 0$ , we denote by  $B_{\varepsilon}(p) \subseteq \mathbb{R}^n$  the closed  $n$ -ball with radius  $\varepsilon$  around  $p$ . Consider the intersection  $B_{\varepsilon}(p) \cap K$ . Since  $K$  is  $n$ -dimensional, we have  $B_{\varepsilon}(p) \cap (K \setminus \partial K) \neq \emptyset$  and if  $\varepsilon$  is sufficiently small, then

$$B_{\varepsilon}(p) \cap \partial K = B_{\varepsilon}(p) \cap \partial K \cap \left( \bigcap_{j=1}^k \partial \mathcal{A}(f_j) \right) = B_{\varepsilon}(p) \cap \left( \bigcap_{j=1}^k \overline{\partial E_{\alpha_j}(f_j)} \right).$$

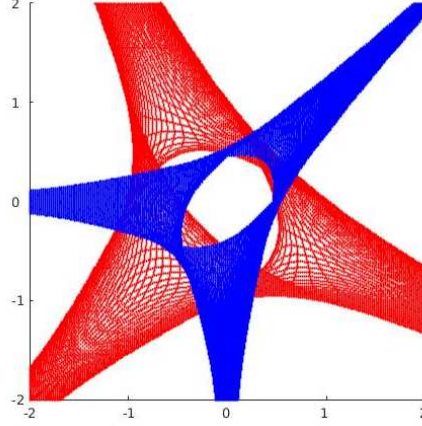


FIGURE 2. The intersection of  $f_1(z_1, z_2) := z_1^2 z_2^2 + z_1 z_2 + z_1 + z_2$  and  $f_2(z_1, z_2) := z_1^3 + z_2^3 + 2z_1 z_2 + 1$ . None of the vertices of  $\mathcal{I}(\{f_1, f_2\})$  given by the intersection of the boundaries of the two bounded components of the complement of  $\mathcal{A}(f_1)$  and  $\mathcal{A}(f_2)$  are vertices of the corresponding intersection polytopes.

Since  $\mathcal{F}$  is generic the intersection  $B_\varepsilon(p) \cap \left( \bigcap_{j=1}^k \overline{\partial E_{\alpha_j}(f_j)} \right)$  is of dimension  $n - k \geq 1$  and contains  $p$  in its interior. By convexity of the components of the complement of an amoeba, there exist points  $p_1, \dots, p_{n-k}, p_{n-k+1} \in B_\varepsilon(p) \cap \partial K$  and  $q \in B_\varepsilon(p) \cap (K \setminus \partial K)$ , spanning an  $(n - k + 1)$ -simplex  $\Gamma$  that contains  $p$  in its interior. If  $\varepsilon$  is sufficiently small, then one has  $\Gamma \subseteq \text{conv}(K)$ . Hence,  $p$  lies in the interior of  $K$ , which is a contradiction since  $p$  was chosen to be an extreme point. Thus,  $k < n$  cannot happen and this implies  $E(K) \subseteq V(K)$ .

We conclude the “particularly”-statement. By the Krein-Milman Theorem [11],  $\text{conv}(K)$  is the convex hull of its extreme points and we hence obtain

$$\text{conv}(K) = \text{conv}(E(K)) \subseteq \text{conv}(V(K)) = P_K.$$

The other inclusion  $P_K \subseteq \text{conv}(K)$  follows directly from  $V(K) \subseteq K$ .

It remains to show that every extreme point of  $\text{conv}(V(K))$  is indeed a vertex of  $P_K$ , i.e.,  $E(K) \subseteq V(P_K)$ . Since  $\text{conv}(K) = P_K = \text{conv}(V(K))$  it follows that  $\text{conv}(K)$  is a polytope. As such its set of extreme points  $E(K)$  and its set of vertices  $V(P_K)$  coincide, which shows the claim.  $\square$

We recall that an  $n$ -dimensional polytope  $P$  is *simple* if every vertex of  $P$  lies in exactly  $n$  facets. Equivalently, every vertex of  $P$  is contained in exactly  $n$  edges. For intersection polytopes the following statement holds.

**Proposition 3.9.** *Let  $\mathcal{F} := \{f_1, \dots, f_n\} \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$  be a generic collection of Laurent polynomials and let  $K$  be a connected component of  $\mathcal{I}(\mathcal{F})$ . Then, the corresponding intersection polytope  $P_K$  is simple.*

*Proof.* Let  $p$  be a vertex of  $P_K$ . Since  $P_K$  is  $n$ -dimensional,  $p$  has to lie in at least  $n$  facets. On the other hand, as  $p$  is a vertex of  $P_K$ , there exist components  $E_{\alpha_1}(f_1), \dots, E_{\alpha_n}(f_n)$  of the complements of the corresponding amoebas such that  $p$  lies on the intersection of their boundaries. Since every facet containing  $p$  has to be determined by one of those components  $E_{\alpha_j}(f_j)$  and since, vice versa, every such component of the complement determines no more than one facet, we conclude that  $p$  lies in at most  $n$  facets. The claim follows.  $\square$

Let  $\mathcal{F} := \{f_1, \dots, f_n\} \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$ . Since a spine is a deformation retract of its corresponding amoeba, it is natural to ask if the different connected components of an intersection  $\mathcal{I}(\mathcal{F})$  can already be detected from the intersection of the spines of  $\mathcal{A}(f_1), \dots, \mathcal{A}(f_n)$ . On the one hand, if the spines have a common intersection, then this gives rise to a connected component of  $\mathcal{I}(\mathcal{F})$ . On the other hand, it is not clear a priori whether every connected component of  $\mathcal{I}(\mathcal{F})$  contains a common point of intersection of the spines of  $\mathcal{A}(f_1), \dots, \mathcal{A}(f_n)$ . To provide an answer to the latter question, we need the following preparatory lemma.

**Lemma 3.10.** *Let  $\mathcal{F} := \{f_1, \dots, f_n\} \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$  be a generic collection of Laurent polynomials and let  $K$  be a connected component of  $\mathcal{I}(\mathcal{F})$ . For every  $1 \leq \ell \leq n$ , there exist  $\alpha, \beta \in \text{New}(f_\ell) \cap \mathbb{Z}^n$ , with  $\alpha \neq \beta$  such that*

$$\overline{E_\alpha(f_\ell)} \cap K \neq \emptyset \quad \text{and} \quad \overline{E_\beta(f_\ell)} \cap K \neq \emptyset.$$

*Proof.* Assume by contradiction that there exists  $f_\ell$  such that  $\overline{E_\alpha(f_\ell)} \cap K \neq \emptyset$  only for a single  $\alpha \in \text{New}(f_\ell) \cap \mathbb{Z}^n$ . Since, by Lemma 3.7, every amoeba  $\mathcal{A}(f_1), \dots, \mathcal{A}(f_n)$  contributes to every vertex of  $K$ , it follows that all vertices of  $K$  lie on  $\partial \overline{E_\alpha(f_\ell)}$ . By convexity of  $E_\alpha(f_\ell)$  we conclude that  $P_K \subseteq \overline{E_\alpha(f_\ell)}$  and thus also  $K \subseteq \overline{E_\alpha(f_\ell)}$ . As  $K \subseteq \mathcal{A}(f_\ell)$  by definition of  $K$ , we infer that  $K \subseteq \partial \overline{E_\alpha(f_\ell)}$ , which implies that  $\dim K \leq n - 1$ . This contradicts Lemma 3.3 (b) and hence the claim follows.  $\square$

Assuming that  $\mathcal{F} := \{f_1, \dots, f_n\} \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$  is a generic collection of Laurent polynomials, we know that the spines of  $\mathcal{A}(f_1), \dots, \mathcal{A}(f_n)$  have a non-empty, finite intersection, i.e.,  $\bigcap_{j=1}^n \mathcal{S}(f_j)$  consists of finitely many points. The next theorem shows that the number of these intersection points already provides an upper bound for the number of connected components of  $\mathcal{I}(\mathcal{F})$

**Theorem 3.11.** *Let  $\mathcal{F} := \{f_1, \dots, f_n\} \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$  be a generic collection of Laurent polynomials and let  $K$  be a connected component of  $\mathcal{I}(\mathcal{F})$ . Then:*

$$K \cap \bigcap_{j=1}^n \mathcal{S}(f_j) \neq \emptyset.$$

*Particularly, the number of connected components of  $\mathcal{I}(\mathcal{F})$  is at most  $\#\bigcap_{j=1}^n \mathcal{S}(f_j)$ .*

*Proof.* Let  $K$  be a connected component of  $\mathcal{I}(\mathcal{F})$ . We will show that for every  $J \subseteq [n]$  the intersection  $\bigcap_{\ell \in J} \mathcal{S}(f_\ell) \cap K$  is non-empty and of dimension  $n - \#J$ .

We proceed by induction on  $\#J$ .

Let  $\#J = 1$ , i.e.,  $J := \{\ell\}$  for some  $1 \leq \ell \leq n$ . By Lemma 3.10, we know that  $K$  intersects the closure of at least two different components  $\overline{E_\alpha(f_\ell)}, \overline{E_\beta(f_\ell)}$  of the complement of  $\mathcal{A}(f_\ell)$  non-trivially. Let  $p$  and  $q$  be points in  $\overline{E_\alpha(f_\ell)}$  and  $\overline{E_\beta(f_\ell)}$ , respectively. Since  $K$  is connected, there exists a path  $\gamma$  from  $p$  to  $q$  inside  $K$ . Since  $\mathcal{S}(f_\ell)$  is a deformation retract of  $\mathcal{A}(f_\ell)$  (see Section 2.3), it follows that  $\mathcal{S}(f_\ell)$  intersects the interior of  $\gamma$ . Moreover, as  $K$  is of dimension  $n$  and as  $\mathcal{S}(f_\ell)$  is a tropical  $(n-1)$ -dimensional hypersurface lying in the interior of  $\mathcal{A}(f_\ell)$ , we further conclude that  $K \cap \mathcal{S}(f_\ell)$  is of dimension  $n-1$ .

Now, assume  $\#J = k \geq 2$ . Without loss of generality, assume  $J = [k]$ . It follows by induction that  $Q := \bigcap_{\ell=1}^{k-1} \mathcal{S}(f_\ell) \cap K$  is non-empty and of dimension  $n - (k-1)$ . Furthermore, assume that  $Q$  is connected (otherwise consider a connected component in the sequel). Being part of the intersection of  $k-1$  tropical hypersurfaces,  $Q$  has to intersect the boundary of  $\partial(K)$ . Moreover, as any spine lies in the interior of its amoeba,  $Q$  cannot intersect facets of  $K$  given by  $\mathcal{A}(f_1), \dots, \mathcal{A}(f_{k-1})$ . Since  $\bigcap_{\ell=1}^{k-1} \mathcal{S}(f_\ell)$  is unbounded but  $Q$  itself is bounded, it follows that  $Q$  has to intersect the boundary of every  $\mathcal{A}(f_k), \dots, \mathcal{A}(f_n)$ . Moreover, using Lemma 3.10, we can further conclude that  $Q$  intersects the boundaries of two different non-empty components,  $E_\alpha(f_k)$  and  $E_\beta(f_k)$  of the complement of  $\mathcal{A}(f_k)$ . Let  $p \in Q \cap \overline{E_\alpha(f_k)}$  and  $q \in Q \cap \overline{E_\beta(f_k)}$ . Since  $Q$  is connected, there exists a path  $\gamma$  connecting  $p$  and  $q$  that lies inside of  $Q$ . As in the case  $\#J = 1$  we can conclude that  $\gamma$  intersects  $\mathcal{S}(f_k)$  and particularly  $Q \cap \mathcal{S}(f_k) \neq \emptyset$ . Since the latter is the intersection of  $k$  generic tropical hypersurfaces with the  $n$ -dimensional closed set  $K$ , we infer that  $\bigcap_{\ell=1}^k \mathcal{S}(f_\ell) \cap K$  is  $(n-k)$ -dimensional; see condition (3) in the definition of generic. This completes the proof of the first part. The ‘‘Particularly’’-statement follows immediately from the first part, since any connected component of  $\mathcal{I}(\mathcal{F})$  contains at least one point of  $\bigcap_{j=1}^n \mathcal{S}(f_j)$ .  $\square$

As an almost immediate consequence of the previous theorem we obtain the Amoeba Bernstein Theorem.

**Amoeba Bernstein Theorem 3.12.** *Let  $\mathcal{F} := \{f_1, \dots, f_n\} \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$  be a generic collection of Laurent polynomials. Let  $c$  be the number of connected components of  $\mathcal{I}(\mathcal{F})$ . Then  $c$  is bounded from above by the number of mixed cells in the subdivision of  $\text{New}(f_1) + \dots + \text{New}(f_n)$ , which is induced by the subdivisions of  $\text{New}(f_1), \dots, \text{New}(f_n)$  that are dual to the spines  $\mathcal{S}(f_1), \dots, \mathcal{S}(f_n)$ . Particularly,  $c$  is bounded from above by the mixed volume  $\text{MV}(\text{New}(f_1), \dots, \text{New}(f_n))$ .*

*Proof.* By Theorem 3.11 we have the inequality  $c \leq \#\bigcap_{j=1}^n \mathcal{S}(f_j)$ . The spines have a stable intersection by our genericity assumption. By the Tropical Bernstein Theorem 2.7 every element of the intersection  $\bigcap_{j=1}^n \mathcal{S}(f_j)$  is dual to a mixed cell in the subdivision of  $\text{New}(f_1) + \dots + \text{New}(f_n)$  which is induced by the subdivisions of  $\text{New}(f_1), \dots, \text{New}(f_n)$  which are pairwise dual to the spines  $\mathcal{S}(f_1), \dots, \mathcal{S}(f_n)$ . By Lemma 2.6  $\text{MV}(\text{New}(f_1), \dots, \text{New}(f_n))$  equals the sum of the volumes of the mixed cells of the induced subdivision in  $\text{New}(f_1) + \dots + \text{New}(f_n)$ . Since our volume form is a lattice volume form induced by  $\mathbb{Z}^n$  every mixed cell has at least volume one and the statement follows.  $\square$



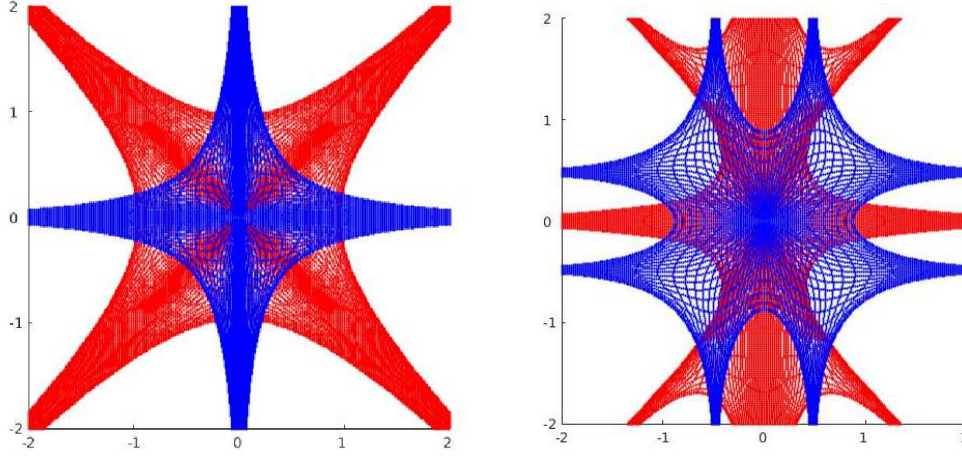


FIGURE 3. *Left picture:* The intersection of  $f_1(z_1, z_2) := z_1^2 z_2^2 + z_1^2 + z_2^2 + 1$  and  $f_2(z_1, z_2) := z_1^3 + z_2^3 + z_1 + z_2$ . *Right picture:* The intersection of  $f_1(z_1, z_2) := z_1^3 z_2^4 + z_1 z_2^4 + 2z_1^4 z_2^3 + 2z_2^3 + 2z_1^4 z_2 + 2z_2 + z_1^3 + z_1$  and  $f_2(z_1, z_2) := z_1^4 z_2^4 + 3z_1^2 z_2^4 + z_2^4 + 3z_1^4 z_2^2 + 3z_2^2 + z_1^4 + 3z_1^2 + 1$ . One can successively go on to construct intersections of amoebas with additional vertices in  $V(\mathcal{F})$ .

As a corollary of Theorem 3.11 we also obtain the following Bézout type statement for the intersection of amoebas.

**Amoeba Bézout Theorem 3.13.** *Let  $\mathcal{F} := \{f_1, \dots, f_n\} \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$  be a generic collection of Laurent polynomials. For  $1 \leq j \leq n$ , let  $d_j := \deg(\text{trop}_S(f_j))$ . Let  $c$  be the number of connected components of  $\mathcal{I}(\mathcal{F})$ . Then,  $c \leq \prod_{j=1}^n d_j$ . Particularly,  $c \leq \prod_{j=1}^n \deg(f_j)$ .*

*Proof.* By Theorem 3.11 we have the inequality  $c \leq \# \bigcap_{j=1}^n \mathcal{S}(f_j)$ . By the tropical Bézout theorem we conclude  $c \leq \prod_{j=1}^n d_j$ ; see e.g., [12]. By construction of the spine we know that  $\mathcal{S}(f_j)$  is dual to a regular subdivision of  $\text{New}(f_j)$  for every  $1 \leq j \leq n$ ; see Section 2.3. Thus,  $c \leq \prod_{j=1}^n \deg(f_j)$ .  $\square$

Though we have just seen that the number of connected components of  $\mathcal{I}(\mathcal{F})$  is bounded by the mixed volume and the degrees of the initial polynomials, it remains an open question, how many vertices a connected component of  $\mathcal{I}(\mathcal{F})$  can have. In particular, it is easy to see that the number of vertices  $V(\mathcal{F})$  of  $\mathcal{I}(\mathcal{F})$  are not bounded by the dimension alone. Indeed, for any  $m \in \mathbb{N}$  one can construct amoebas such that there exist components of the complement having  $2m$  vertices. Figure 3 shows two such examples for  $m = 4$  and  $m = 8$ .

#### 4. A GENERALIZED ORDER MAP

In this section, we generalize the order map of an amoeba (see Section 2.2) to intersections of generic collections of amoebas. So as to obtain a well-defined map it is necessary to assume that if a polynomial  $f$  belongs to a generic collection  $\mathcal{F} \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$  of Laurent

polynomials, then for any point  $p$  on the boundary of  $\mathcal{A}(f)$  there exists a *unique* component of the complement of  $\mathcal{A}(f)$  containing  $p$  in its closure; see condition (4) in our definition of “generic”. We remark that this is not for all boundary points of all amoebas the case; see [22, Figure 2, p. 58] for an example. However, if  $f$  is generic, then the order map can be extended to the boundary of an amoeba. Namely, if  $p \in \overline{E_\alpha(f)} \cap \partial\mathcal{A}(f)$ , then we define  $\text{ord}(p) := \alpha$ , i.e., the order of a point  $p$  on the boundary of an amoeba  $\mathcal{A}(f)$  is the order of the unique component of the complement, which contains  $p$  in its closure. From now on assume that  $\mathcal{F} := \{f_1, \dots, f_n\} \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$  is a generic collection of Laurent polynomials. For  $1 \leq i \leq n$ , we use  $\text{ord}_i$  to denote the just described extension of the order map of  $f_i$ . In order to simplify the notation further, we write  $\text{New}(\mathcal{F})$  for the Cartesian product of the Newton polytopes of  $f_1, \dots, f_n$ , i.e.,

$$\text{New}(\mathcal{F}) := \text{New}(f_1) \times \dots \times \text{New}(f_n).$$

Given these prerequisites, we generalize the order map of amoebas to intersections of amoebas in the following way: The *generalized order map* of  $\mathcal{F}$  is defined by

$$\text{ord}_{\mathcal{F}} : V(\mathcal{F}) \rightarrow \mathbb{Z}^{n \times n} \quad p \mapsto \begin{pmatrix} \text{ord}_1(p) \\ \vdots \\ \text{ord}_n(p) \end{pmatrix}.$$

We refer to  $\text{ord}_{\mathcal{F}}(p)$  as the *order matrix* of  $p \in V(\mathcal{F})$ .

Though the generalized order map  $\text{ord}_{\mathcal{F}}$  is not injective in general, the next theorem shows that this is indeed the case if one restricts to a single intersection polytope.

**Theorem 4.1.** *Let  $\mathcal{F} := \{f_1, \dots, f_n\} \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$  be a generic collection of Laurent polynomials. Let further  $K$  be a connected component of  $\mathcal{I}(\mathcal{F})$  and let  $P_K$  be the corresponding intersection polytope. Let  $p$  and  $q$  be vertices of  $P_K$ . Then*

$$\text{ord}_{\mathcal{F}}(p) \neq \text{ord}_{\mathcal{F}}(q).$$

*Particularly,  $\text{ord}_{\mathcal{F}}$  is injective on  $V(P_K)$ .*

*Proof.* In order to prove injectivity of the restriction of  $\text{ord}_{\mathcal{F}}$  to  $V(P_K)$ , we show that for vertices  $p, q \in V(P_K)$  there exists an  $l$  with  $1 \leq l \leq n$  and  $\text{ord}_l(p) \neq \text{ord}_l(q)$ . Assume by contradiction that  $\text{ord}(p) = \text{ord}(q) =: (\alpha(1), \dots, \alpha(n))$ , i.e.,  $\text{ord}_i(p) = \text{ord}_i(q) = \alpha(i)$  for  $1 \leq i \leq n$ . Consider the line segment  $\sigma$  between  $p$  and  $q$  and let  $v := q - p$  be the corresponding vector pointing from  $p$  to  $q$ . By assumption, Definition 3.5 and Theorem 3.8, we have  $p, q \in \overline{E_{\alpha(i)}(f_i)}$  for  $1 \leq i \leq n$  and, by convexity of  $E_{\alpha(i)}(f_i)$ , we conclude that  $\sigma \subseteq \overline{E_{\alpha(i)}(f_i)}$  for  $1 \leq i \leq n$ . Hence,  $q - \varepsilon \cdot v \in \overline{E_{\alpha(i)}(f_i)}$  for  $1 \leq i \leq n$  and  $0 \leq \varepsilon \leq 1$ . Moreover, since  $q$  lies on the boundary of  $\mathcal{A}(f_i)$  for  $1 \leq i \leq n$  and  $\overline{E_{\alpha(i)}(f_i)}$  is strictly convex, we infer that  $q + \varepsilon \cdot v \in \mathcal{A}(f_i)$  for  $\varepsilon > 0$  sufficiently small and  $1 \leq i \leq n$ . Particularly,  $q + \varepsilon \cdot v \in \mathcal{I}(\mathcal{F})$  for  $\varepsilon > 0$  sufficiently small, i.e.,  $q + \varepsilon \cdot v \in P_K$  for  $\varepsilon > 0$  sufficiently small. Hence,  $q$  cannot be a vertex of  $P_K$  and we obtain a contradiction.  $\square$

Though, by Theorem 4.1 the generalized order map is injective on each intersection polytope, this does not have to be true if one considers restrictions of the generalized

order map to the vertex set of a connected component of  $\mathcal{I}(\mathcal{F})$ . For instance, looking at Figure 2 one sees that both connected components have two vertices, that are not vertices of the corresponding intersection polytopes, whose orders are equal. Namely, they equal the order of the two bounded components of the complements of the amoebas. It also follows from the proof of Theorem 4.1 that this example describes the only possible other case. More precisely, if  $K$  is a connected component of  $\mathcal{I}(\mathcal{F})$  and if  $\text{ord}(p) = \text{ord}(q)$  for two vertices  $p, q \in V(K)$ , then neither  $p$  nor  $q$  can be vertices of the corresponding intersection polytope  $P_K$ .

The next proposition describes the normal cones of specific vertices of an intersection polytope.

**Proposition 4.2.** *Let  $\mathcal{F} := \{f_1, \dots, f_n\} \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$  be a generic collection of Laurent polynomials. Let  $K$  be a connected component of  $\mathcal{I}(\mathcal{F})$  and  $P_K$  be the corresponding intersection polytope. Let  $p \in V(P_K)$  be a vertex of  $P_K$  such that  $\text{ord}_i(p)$  corresponds to a vertex  $v_i$  of  $\text{New}(f_i)$  for  $1 \leq i \leq n$ . Then the normal cone of  $p$  in  $P_K$  contains an affine translation of the intersection of all normal cones of the  $v_i$  in  $\text{New}(f_i)$  for  $1 \leq i \leq n$ , i.e., there exists a  $v \in \mathbb{R}^n$  such that*

$$v + \bigcap_{i=1}^n \text{NF}_{v_i}(f_i) \subseteq \text{NF}_p(P_K).$$

*Proof.* By assumption and Theorem 4.1, the vertex  $p$  is the unique vertex of  $P_K$ , given by the intersection of  $\overline{E_{v_i}(f_i)}$  for  $1 \leq i \leq n$ . As  $v_i$  is a vertex of  $\text{New}(f_i)$ , Theorem 2.2 implies, that the component  $E_{v_i}(f_i)$  of the complement of  $\mathcal{A}(f_i)$  contains an affine translation of the normal cone  $\text{NF}_{v_i}(f_i)$ . Hence, the intersection  $\bigcap_{i=1}^n E_{v_i}(f_i)$  contains an affine translation of  $\bigcap_{i=1}^n \text{NF}_{v_i}(f_i)$ . Moreover, the normal cone  $\text{NF}_p(P_K)$  has to contain all points given by  $p + q$ , that are not contained in any of the amoebas  $\mathcal{A}(f_i)$ , i.e.,  $\bigcap_{i=1}^n E_{v_i}(f_i) \subseteq \text{NF}_p(P_K)$ . Thus, we can conclude  $v + \bigcap_{i=1}^n \text{NF}_{v_i}(f_i) \subseteq \text{NF}_p(P_K)$  for some  $v \in \mathbb{R}^n$ .  $\square$

Since there is a relation between components of the complement of a single amoeba and the vertices of the corresponding Newton polytope (see Theorem 2.2) it is reasonable to ask if such a relation also exists if one considers intersections of amoebas and the Minkowski sum of their Newton polytopes. Before we can provide such a relation, we need to introduce some further notion.

For given Newton polytopes  $\text{New}(f_1), \dots, \text{New}(f_n)$  one defines the *common refinement*  $\text{NF}(f_1, \dots, f_n)$  of the normal fans  $\text{NF}(f_1), \dots, \text{NF}(f_n)$  as the fan given by all cones of the form  $\bigcap_{j=1}^n \text{NF}_{G_j}(f_j)$  where  $\text{NF}_{G_j}(f_j)$  is an arbitrary cone of  $\text{NF}(f_j)$ .

**Definition 4.3.** *We call a non-empty, full dimensional cone  $\bigcap_{j=1}^n \text{NF}_{G_j}(f_j)$  in the common refinement  $\text{NF}(f_1, \dots, f_n)$  *mixed* if it satisfies for all  $1 \leq i \leq n$ :*

$$\text{NF}_{G_i}(f_i) \cap \bigcap_{j \in [n] \setminus \{i\}} \text{NF}_{G_j}(f_j) \neq \text{NF}_{G_i}(f_i), \quad \bigcap_{j \in [n] \setminus \{i\}} \text{NF}_{G_j}(f_j).$$

In the following we make use of the well-known fact [25, Proposition 7.12.] that the normal fan of a Minkowski sum  $\text{New}(f_1) + \text{New}(f_2)$  equals the common refinement of the individual normal fans  $\text{NF}(f_1)$  and  $\text{NF}(f_2)$ .

Since we are aiming at a possible connection between  $\mathcal{I}(\mathcal{F})$  and the Minkowski sum of  $\text{New}(f_1), \dots, \text{New}(f_n)$ , it is reasonable to consider not only a single intersection polytope but the convex hull of all intersection polytopes. In other words, we are interested in  $\text{conv}(V(\mathcal{F}))$ . It is an obvious question, which points in  $V(\mathcal{F})$  are actually vertices of  $\text{conv}(V(\mathcal{F}))$ . We are able to provide a characterization of the vertex set of  $\text{conv}(V(\mathcal{F}))$  in terms of vertices of the Minkowski sum  $\text{New}(f_1) + \dots + \text{New}(f_n)$  alone. To simplify the notation we write  $\text{Mink}(\mathcal{F})$  for this Minkowski sum in the following.

**Theorem 4.4.** *Let  $\mathcal{F} := \{f_1, \dots, f_n\} \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$  be a generic collection of Laurent polynomials. Let  $p \in V(\mathcal{F})$ . Then the following are equivalent:*

- (a)  *$p$  is a vertex of  $\text{conv}(V(\mathcal{F}))$ .*
- (b) *For  $1 \leq j \leq n$  there exists a vertex  $v := v_1 + \dots + v_n \in \text{Mink}(\mathcal{F})$ , where  $v_j \in \text{New}(f_j)$  is a vertex, such that  $p \in \bigcap_{j=1}^n \overline{\partial(E_{v_j}(f_j))}$  and  $\text{NF}_v(\text{Mink}(\mathcal{F}))$  is a mixed cone.*

We remark that it will follow from the proof of the Theorem 4.4 that in the situation of (b), the intersection  $\bigcap_{j=1}^n \overline{\partial(E_{v_j}(f_j))}$  equals a single point, which is a vertex of  $\text{conv}(V(\mathcal{F}))$ .

*Proof.* Let  $P$  denote the polytope that is given as the convex hull of  $V(\mathcal{F})$ , i.e.,  $P := \text{conv}(V(\mathcal{F}))$ . Note that, by construction, the set of vertices of  $P$  is a subset of  $V(\mathcal{F})$ .

We first show “(b)  $\Rightarrow$  (a)”:

Let  $v := v_1 + \dots + v_n \in \text{Mink}(\mathcal{F})$  be a vertex such that  $v_j$  is a vertex of  $\text{New}(f_j)$  for  $1 \leq j \leq n$  and  $C := \text{NF}_v(\text{Mink}(\mathcal{F}))$  is a full-dimensional mixed cone in the common refinement  $\text{NF}(\mathcal{F}) := \text{NF}(f_1, \dots, f_n)$ . Then it holds that  $C = \bigcap_{j=1}^n \text{NF}_{v_j}(f_j)$ . By Theorem 2.2 it follows furthermore that  $\bigcap_{j=1}^n \overline{\partial E_{v_j}(f_j)} \neq \emptyset$ . Since  $C$  is mixed, we can conclude that  $\bigcap_{j=1}^n \overline{\partial E_{v_j}(f_j)}$  has to be a single point. Indeed, for  $n = 2$ , this is obvious, since  $\overline{\partial E_{v_1}(f_1)} \cap \overline{\partial E_{v_2}(f_2)} \neq \emptyset$  and  $\text{NF}_{v_1}(f_1) \cap \text{NF}_{v_2}(f_2) \neq \text{NF}_{v_1}(f_1), \text{NF}_{v_2}(f_2)$ . For arbitrary  $n$ , the claim follows from the same argument applied to every  $\overline{\partial E_{v_i}(f_i)}$ , the one-dimensional subset given by  $\bigcap_{j \in [n] \setminus \{i\}} \overline{\partial E_{v_j}(f_j)}$ , and the corresponding cones with the assumption that  $C$  is mixed. Hence,  $\bigcap_{j=1}^n \overline{\partial E_{v_j}(f_j)}$  equals a point  $p$ . By Theorem 2.2 an affine translation of  $C$  is contained in  $\bigcap_{j=1}^n \overline{E_{v_j}(f_j)}$  and therefore  $C$  cannot intersect  $P$  in any point but  $p$ . Thus, we have  $(p + C) \cap P = p$  and hence  $p$  is a vertex of  $P$ .

It remains to show “(a)  $\Rightarrow$  (b)”.

Let  $p$  be a vertex of  $P$ . It follows from Lemma 3.7 that there exist components  $E_{\alpha_1}(f_1), \dots, E_{\alpha_n}(f_n)$  of the complements of the amoebas  $\mathcal{A}(f_1), \dots, \mathcal{A}(f_n)$  such that  $p \in \bigcap_{j=1}^n \overline{\partial E_{\alpha_j}(f_j)}$ . We prove the claim by contradiction. First assume that  $\alpha_1, \dots, \alpha_n$  are vertices of  $\text{New}(f_1), \dots, \text{New}(f_n)$  but  $\text{NF}_v(\text{Mink}(\mathcal{F}))$  is not a mixed cone for  $v := \alpha_1 + \dots + \alpha_n$ . Hence, there exist  $1 \leq i \leq n$  such that  $\text{NF}_{\alpha_i}(f_i) \subseteq \bigcap_{j \in [n] \setminus \{i\}} \text{NF}_{\alpha_j}(f_j)$ . Thus, two possible cases concerning the behavior of the corresponding components of the amoebas complements may occur:

**Case 1:**  $\overline{E_{\alpha_i}(f_i)} \subset \bigcap_{j \in [n] \setminus \{i\}} \overline{E_{\alpha_j}(f_j)}$ . But then  $p$  cannot be a vertex of  $P$  since, by Lemma 3.7, all the  $f_i$  need to be distinct, so as to obtain a vertex as the intersection of the boundary of  $n$  non-redundant components  $\overline{E_{\alpha_j}(f_j)}$ .

**Case 2:**  $\overline{E_{\alpha_i}(f_i)} \setminus \bigcup_{j \in [n] \setminus \{i\}} \overline{\partial E_{\alpha_j}(f_j)}$  consists of two connected components  $S_1, S_2$  satisfying that  $S_1$  is contained in  $\bigcap_{j \in [n] \setminus \{i\}} E_{\alpha_j}(f_j)$  and  $S_2$  is contained in the complement of  $\bigcap_{j \in [n] \setminus \{i\}} \overline{E_{\alpha_j}(f_j)}$ .

Consider the set  $T := \bigcap_{j \in [n] \setminus \{i\}} \overline{\partial E_{\alpha_j}(f_j)}$ . Assume first that  $S_2 = \emptyset$ , i.e.,  $\overline{E_{\alpha_i}(f_i)} \subseteq \bigcap_{j \in [n] \setminus \{i\}} \overline{E_{\alpha_j}(f_j)}$  and  $\partial \overline{E_{\alpha_i}(f_i)}$  intersects  $T$  in a single point. Since this implies  $E_{\alpha_i}(f_i) \subset \bigcap_{j \in [n] \setminus \{i\}} \overline{E_{\alpha_j}(f_j)}$ , we conclude that  $T$  intersects the boundaries of two components of the complement of  $\mathcal{A}(f_i)$  which are distinct from  $E_{\alpha_i}(f_i)$ . Since boundaries of such components are of codimension 1 and  $T$  is of codimension  $n - 1$  this intersection yield points  $p_1, p_2$ , which are contained in  $P$ . Since we have  $T \subseteq \overline{\partial E_{\alpha_j}(f_j)}$  for every  $j \neq i$  and  $\overline{\partial E_{\alpha_j}(f_j)}$  is strictly convex, the subset of  $T$  connecting  $p_1$  and  $p_2$  is contained in the interior of  $P$ . And since  $E_{\alpha_i}(f_i) \subset \bigcap_{j \in [n] \setminus \{i\}} \overline{E_{\alpha_j}(f_j)}$  it follows that  $p$  is contained in the interior of  $P$ . Thus,  $p$  is not a vertex.

Now, assume  $S_2 \neq \emptyset$ . By construction of Case 2 we have  $S_2 \subseteq \text{int}(P)$ ,  $S_1 \not\subseteq P$ , and  $\partial \overline{E_{\alpha_i}(f_i)}$  intersects  $T$  in two points  $u$  and  $w$ . Hence, both  $u$  and  $w$  need to be vertices of  $P$  or we can argument as in the  $S_2 = \emptyset$  case. Since both  $u$  and  $w$  are given by the intersection of the same components of the complements of  $\mathcal{A}(f_i)$  with  $1 \leq i \leq n$  we have  $\text{ord}(u) = \text{ord}(w)$ . But that is impossible with an analog argument as in the proof of Theorem 4.1.

Finally, assume that  $p \in \bigcap_{j=1}^n \overline{\partial E_{\alpha_j}(f_j)}$  is a vertex of  $P$  and at least one  $\overline{E_{\alpha_i}(f_i)}$  does not correspond to a vertex of  $\text{New}(f_j)$ . Namely, in this case we can consider the same set  $T$  as above and by convexity of  $\overline{E_{\alpha_i}(f_i)}$  the intersection  $T \cap \partial \overline{E_{\alpha_i}(f_i)}$  is of cardinality two. Moreover, both points of intersection have the same order, which, as in the proof of Theorem 4.1, again yields a contradiction.  $\square$

We remark that if  $p \in \text{conv}(V(\mathcal{F}))$  is a vertex, then by the proof of Theorem 4.4 we do not only have  $p \in \bigcap_{j=1}^n \overline{\partial E_{v_j}(f_j)}$  with  $v_j$  as in Theorem 4.4 (b), but that it even holds that  $p = \bigcap_{j=1}^n \overline{\partial E_{v_j}(f_j)}$ . Using this fact, the following is an immediate consequence of Theorem 4.4.

**Corollary 4.5.** *Let  $\mathcal{F} := \{f_1, \dots, f_n\} \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$  be a generic collection of Laurent polynomials. Let  $P$  be the convex hull of the vertices of  $\mathcal{I}(\mathcal{F})$ , i.e.,  $P := \text{conv}(V(\mathcal{F}))$  and let  $V(P)$  be the vertex set of  $P$ . Then, the restriction of the order map  $\text{ord}_{\mathcal{F}}$  to  $V(P)$  is injective.*

*Proof.* Let  $p$  be a vertex of  $P$ . Then, as remarked above, Theorem 4.4 implies that there exist vertices  $v_j \in \text{New}(f_j)$ ,  $1 \leq j \leq n$ , such that  $p = \bigcap_{j=1}^n \overline{\partial E_{v_j}(f_j)}$ . Since  $\text{ord}_{\mathcal{F}}(p) = (v_1, \dots, v_n)$  the claim follows.  $\square$

In the following, we associate another canonical polytope to a generic collection of Laurent polynomials  $\mathcal{F} := \{f_1, \dots, f_n\}$ . Let  $\mathcal{V}$  denote the image of the vertex set of  $\mathcal{I}(\mathcal{F})$  under the order map, i.e.,  $\mathcal{V} := \text{ord}_{\mathcal{F}}(V(\mathcal{F}))$ . We call the convex hull of  $\mathcal{V}$  the *order polytope* associated to  $\mathcal{F}$ . We denote this polytope by  $\mathcal{O}(\mathcal{F})$ .

We remark that if one considers generic collections  $\mathcal{F} := \{f_1, \dots, f_k\} \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$  of  $k < n$  Laurent polynomials, then one can generalize the order map in a similar way as above.

Indeed, if a point  $p$  belongs to the intersection of the boundaries of  $\mathcal{A}(f_1), \dots, \mathcal{A}(f_k)$ , then, as before, one finds unique components  $E_{\alpha_1}(f_1), \dots, E_{\alpha_k}(f_k)$  of amoeba complements such that  $p$  lies in their closure. One can now define the order matrix of  $p$  to be the matrix with rows  $\alpha_1, \dots, \alpha_k$ . In this case the order map is a map from  $\bigcap_{i=1}^k \partial \mathcal{A}(f_i)$  to  $\mathbb{Z}^{k \times n}$ . Moreover, one can then define the order polytope of  $\mathcal{F} := \{f_1, \dots, f_k\}$  as the convex hull of the image of the generalized order map. Since this generalization, however, is irrelevant for our purposes, we do not pursue this direction further and postpone it for further study. The following is an easy example of an order polytope in the case that  $k < n$ .

**Example 4.6.** *Consider the trivial case  $\mathcal{F} := \{f\}$ . Then  $\mathcal{O}(\mathcal{F})$  is the convex hull of the image of the usual order map  $\text{ord}$  applied to  $\mathbb{R}^n \setminus \mathcal{A}(f)$ . By Theorem 2.1  $\mathcal{O}(\mathcal{F})$  is a lattice polytope contained in  $\text{New}(f)$ . And by Theorem 2.2 it follows that  $\mathcal{O}(\mathcal{F}) = \text{New}(f)$ .*

If  $\mathcal{F}$ , however, is a collection of more than one Laurent polynomial, then  $\mathcal{O}(\mathcal{F})$  becomes less trivial. We provide some initial statements about the general case.

**Theorem 4.7.** *Let  $\mathcal{F} := \{f_1, \dots, f_n\} \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$  be a generic collection of Laurent polynomials. Then:*

- (a)  *$\mathcal{O}(\mathcal{F})$  is a lattice polytope and contained in  $\text{New}(f_1) \times \dots \times \text{New}(f_n)$ .*
- (b) *Let  $k$  denote the number of mixed cones in the common refinement  $\text{NF}(f_1, \dots, f_n)$ . Then  $\mathcal{O}(\mathcal{F})$  and  $\text{New}(\mathcal{F})$  have at least  $k$  vertices in common. Let  $\text{ord}_{\mathcal{F}}(p)$  be one of these joint vertices and let  $\text{NF}_{\text{ord}_{\mathcal{F}}(p)}(\text{New}(\mathcal{F}))$  be its corresponding normal cone in  $\text{New}(\mathcal{F})$ . Then  $\text{NF}_{\text{ord}_{\mathcal{F}}(p)}(\text{New}(\mathcal{F}))$  contains a real  $(n \times n)$ -matrix in the interior with all rows equal.*

*Proof.* (a) Since  $\text{ord}_{\mathcal{F}}(p) \in \mathbb{Z}^{n \times n}$  Theorem 2.1 implies that  $\mathcal{O}(\mathcal{F})$  is a lattice polytope. Consider a point  $p \in V(\mathcal{F})$ . Theorem 2.1 implies that  $\text{ord}_i(p) \in \text{New}(f_i)$  for  $1 \leq i \leq n$  and hence  $\text{ord}_{\mathcal{F}}(p) \in \text{New}(\mathcal{F})$ . Since, as a polytope,  $\text{New}(\mathcal{F})$  is convex,  $\mathcal{O}(\mathcal{F})$  is contained in  $\text{New}(f_1) \times \dots \times \text{New}(f_n)$ .

(b) By Theorem 4.4 and the succeeding remark, there exists a bijection between the mixed cones in the common refinement  $\text{NF}(f_1, \dots, f_n)$  and the vertices of  $\text{conv}(V(\mathcal{F}))$ . Let  $p := \bigcap_{j=1}^n \overline{E_{\alpha_j}(f_j)}$  be a vertex of  $\text{conv}(V(\mathcal{F}))$ . By Theorem 4.4 (b) every  $\alpha_j$  is a vertex in  $\text{New}(f_j)$ . Thus,  $\text{ord}_{\mathcal{F}}(p) = (\alpha_1, \dots, \alpha_n)$  is a vertex of  $\text{New}(f_1) \times \dots \times \text{New}(f_n)$ . Since  $\mathcal{O}(\mathcal{F}) \subset \text{New}(f_1) \times \dots \times \text{New}(f_n)$  and  $\text{ord}_{\mathcal{F}}(p) \in \mathcal{O}(\mathcal{F})$ , it follows that  $\text{ord}_{\mathcal{F}}(p)$  is also a vertex of  $\mathcal{O}(\mathcal{F})$ . Therefore,  $\mathcal{O}(\mathcal{F})$  and  $\text{New}(f_1) \times \dots \times \text{New}(f_n)$  have at least  $k$  vertices in common.

Furthermore, by Theorem 4.4, the cone  $\text{NF}_{\alpha_1 + \dots + \alpha_n}(\text{Mink}(\mathcal{F}))$  corresponding to  $p$  is a mixed cone. Particularly, the intersection of the normal cones  $\text{NF}_{\alpha_1}(f_1), \dots, \text{NF}_{\alpha_n}(f_n)$  is a full-dimensional cone. Thus, there exists a vector  $(v_1, \dots, v_n) \in \text{int}(\bigcap_{j=1}^n \text{NF}_{\alpha_j}(f_j))$ . By construction the normal cone  $\text{NF}_{\text{ord}(p)}(\text{New}(\mathcal{F}))$  is the Cartesian product of the normal cones  $\text{NF}_{\alpha_1}(f_1), \dots, \text{NF}_{\alpha_n}(f_n)$ . Hence,  $(v_1, \dots, v_n) \in \text{int}(\bigcap_{j=1}^n \text{NF}_{\alpha_j}(f_j))$  if and only if

$$\begin{pmatrix} v_1 & \cdots & v_n \\ \vdots & \ddots & \vdots \\ v_1 & \cdots & v_n \end{pmatrix} \in \text{int}(\text{NF}_{\text{ord}(p)}(\text{New}(\mathcal{F}))).$$



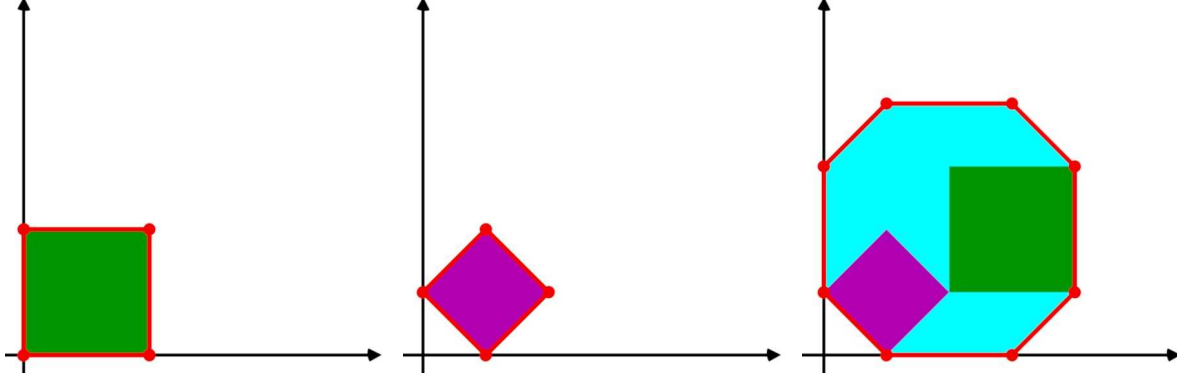


FIGURE 4. The Newton polytopes of  $f_1(z_1, z_2) := z_1^2 z_2^2 + z_1^2 + z_2^2 + 1$  and  $f_2(z_1, z_2) := z_1^3 + z_2^3 + z_1 + z_2$  and their Minkowski sum.

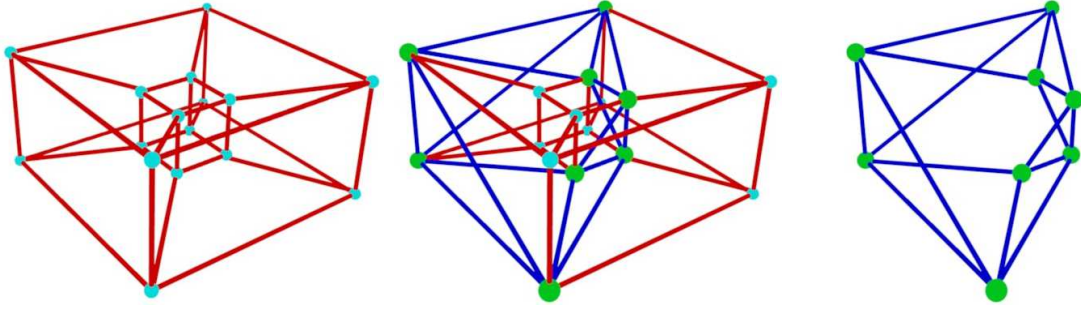


FIGURE 5. The Schlegel diagram of  $\text{New}(f_1) \times \text{New}(f_2)$  with  $f_1, f_2$  as in Figure 4 together with the order polytope  $\mathcal{O}(\mathcal{F})$  for  $\mathcal{F} := \{f_1, f_2\}$  both inside  $\text{New}(f_1) \times \text{New}(f_2)$  and without  $\text{New}(f_1) \times \text{New}(f_2)$ .

□

**Example 4.8.** We consider once more the intersection  $\mathcal{I}(\mathcal{F})$  with  $\mathcal{F}$  given by  $f_1(z_1, z_2) := z_1^2 z_2^2 + z_1^2 + z_2^2 + 1$  and  $f_2(z_1, z_2) := z_1^3 + z_2^3 + z_1 + z_2$ .  $\mathcal{I}(\mathcal{F})$  is shown in the left picture of Figure 3;  $V(\mathcal{F})$  has 8 elements, which are all vertices in  $\text{conv}(V(\mathcal{F}))$ . Hence, all elements correspond to a unique vertex in the Minkowski sum; see Figure 4.  $\text{New}(f_1) \times \text{New}(f_2)$  is a polytope with 16 vertices given by the product of two rectangles in  $\mathbb{R}^4$ . It follows that  $\mathcal{O}(\mathcal{F})$  is the convex hull of 8 of these 16 vertices as shown in the Schlegel diagram in Figure 5.

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